HOW TO CALCULATE MASS PROPERTIES
(An Engineer's Practical Guide)

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Abstract There are numerous text books on dynamics which devote a few pages to the calculation of mass properties. However, these text books quickly jump from a very brief description of these quantities to some general mathematical formulas without giving adequate examples or explaining in enough detail how to use these formulas. The purpose of this paper is to provide a detailed procedure for the calculation of mass properties for an engineer who is inexperienced in these calculations. Hopefully this paper will also provide a convenient reference for those who are already familiar with this subject.

This paper contains a number of specific examples with emphasis on units of measurement. The examples used are rockets and re-entry vehicles. The paper then describes the techniques for combining the mass properties of sub-assemblies to yield the composite mass properties of the total vehicle. Errors due to misalignment of the stages of a rocket are evaluated numerically. Methods for calculating mass property corrections are also explained.

Calculation of Mass Properties using Traditional Methods

Choosing the Reference Axes

The first step in calculating mass properties of an object is to assign the location of the reference axes. The center of gravity and the product of inertia of an object can have any numerical value or polarity, depending on the choice of axes that are used as a reference for the calculation. Stating that a CG coordinate is "0.050 inches" means nothing unless the position of the reference axis is also precisely defined. Any reference axes may be chosen. For example, the center of gravity of a cylinder may be 4.050 inches from one end, 0.050 inches from its midpoint, and 3.950 inches from the other end. Furthermore, each end of the cylinder may not be perpendicular to the central axis, so that a means of determining the "end" of the cylinder would have to be further defined.

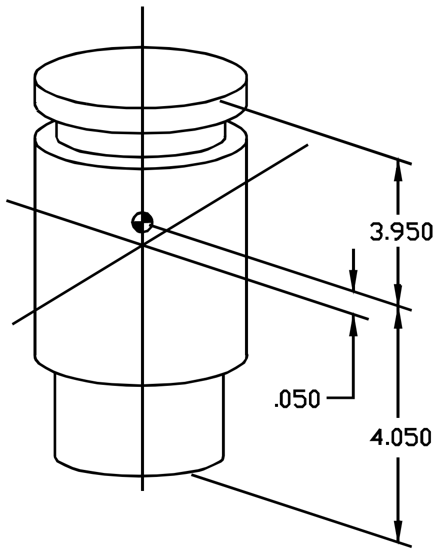


Figure 1 Center of gravity (and product of inertia) are defined relative to orthogonal axes

Three mutually perpendicular reference axes are required to define the location of the center of gravity of an object. These axes are usually selected to coincide with edges of the object, accurately located details, or the geometric center of the object.

It is not sufficient to state that an axis is the centerline of the object. You must also specify which surfaces on the object define this centerline.

Moment of inertia is a rotational quantity and requires only one axis for its reference. Although this can theoretically be any axis in the vicinity of the object, this axis usually is the geometric center, the rotational center (if the object revolves on bearings), or a principal axis (axis passing through the center of gravity which is chosen so the products of inertia are zero).

Product of inertia requires three mutually perpendicular reference axes. One of these axes may be a rotational axis or a geometric centerline.

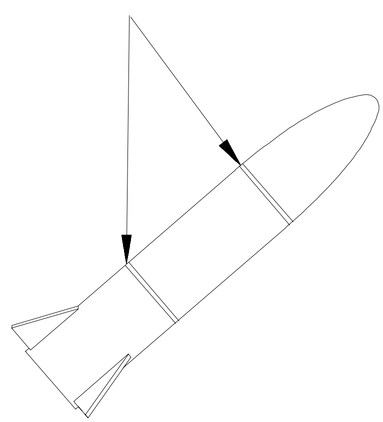


Figure 2 - Datum Rings

For maximum accuracy, it is important to use reference axes that can be located with a high degree of precision. If the object is an aerospace item, then we recommend that this object be designed with two reference datum rings per section, which can be used to define the reference axes. These rings can be precision attachment points that are used to interface the object with another section of a spacecraft or rocket, or they can be rings that were provided solely for the purpose of alignment and/or measurement of mass properties. The accuracy of calculation (and the subsequent accuracy of measurement of an actual piece of hardware) is only as good as the accuracy of the means of locating the reference axes. We have found that the single largest source of error in mass properties calculations is the uncertainty of the reference. The dimensional data provided to the mass properties engineer must be sufficiently accurate to permit mass properties tolerances to be met.

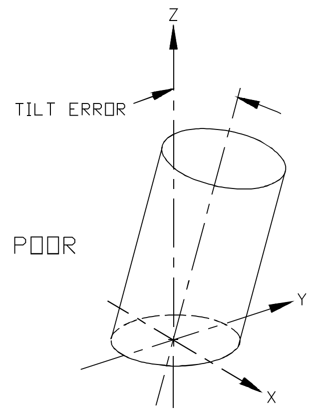


Figure 3

For example, if you are asked to make precise calculations of mass properties of a projectile, you should establish the error due to reference misalignment as the first step in your calculations. If you are required to calculate CG within an accuracy of 0.001 inch and the reference datum is not round within 0.003 inch, then you cannot accomplish your task. There is no sense in making a detailed analysis of the components of an object when the reference error prevents accurate calculations. Furthermore, it will be impossible to accurately measure such a part after it is fabricated and verify the accuracy of your calculations. The location and accuracy of the reference axes must be of the highest precision.

If your task is to calculate the mass properties of a vehicle that is assembled in sections, then serious thought should be given to the accuracy of alignment of the sections when they are assembled. Often this can be the biggest single factor in limiting the degree of balance (if the vehicle was balanced in sections because the total vehicle is too big for the balancing machine). Alignment error is amplified for long rockets . . . a 0.001 inch lean introduced by alignment error on a 12 inch diameter can result in a 0.007 inch CG error on a 15 foot long rocket section. This is discussed in detail in the sections of this paper that present the math for combining the mass properties of subassemblies.

The accuracy required for various types of calculations is summarized in later sections of this paper.

The first step in calculating mass properties is to establish the location of the X, Y, and Z axes. The accuracy of the calculations (and later on the accuracy of the measurements to verify the calculations) will depend entirely on the wisdom used in choosing the axes. Theoretically, these axes can be at any location relative to the object being considered, provided the axes are mutually perpendicular. However, in real life, unless the axes are chosen to be at a location that can be accurately measured and identified, the calculations are meaningless.

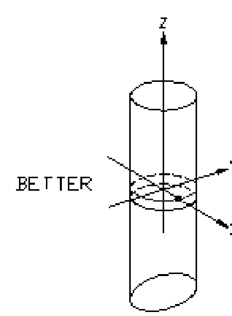


Figure 4

Choosing the Location of the Axes

The axes in Figure 3 do not make a good reference because a small error in squareness of the bottom of the cylinder causes the object to lean away from the vertical axis. The axes below (Figure 4) make a better choice.

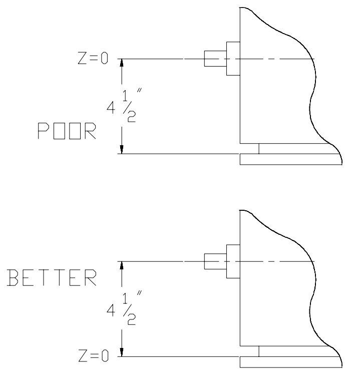


Figure 5

Reference axes must be located at physical points on the object that can be accurately measured. Although the center line of a ring may exist in midair, it can be accurately measured and is therefore a good reference location as can the center of a close tolerance hole which could be identified as the zero degree reference to identify the X axis (Fig. 4).

An axis should always pass through a surface that is rigidly associated with the bulk of the object. In Figure 5 it would be better to locate the origin at the end of the object rather than the fitting that is loosely dimensioned relative to the end.

Calculating CG Location

General Discussion

The center of gravity of an object is:

- also called the "center of mass" of the object.
- the point where the object would balance if placed on a knife edge
- the single point where the static balance moments about three mutually perpendicular axes are all zero.
- the centroid of the volume of the object, if the object is homogeneous.
- the point where all the mass of the object could be considered to be concentrated when performing static calculations.
- the point about which the object rotates in free space
- the point through which the force of gravity can be considered to act
- the point at which an external force must be applied to produce pure translation of an object in space

Center of gravity location is expressed in units of length along each of the three axes (X, Y, and Z). These are the three components of the vector distance from the origin of the coordinate system to the CG location. Center of gravity of composite masses is calculated from moments taken about the origin. The fundamental dimensions of moment are typically Force times Distance. Alternatively, Mass moment may be used with any units of Mass times Distance. For homogeneous elements, volume moments may also be used. Care must be taken to be sure that moments for all elements are expressed in compatible units.

When combining mass elements, a useful technique is to use "offset moments" about each of the three orthogonal axes. The X offset moment of one element (such as $M_{X1} = +3W_1$) can be easily added to the X offset moments of other elements of mass, the sum divided by the total weight, and the result will be the X component of the CG location of the composite mass. Likewise, the Y and Z offset moments ($M_{Y1}=-5W_1$, and $M_{Z1}=+7W_1$) can be combined with similar Y and Z offset moments of other elements to determine the Y and Z components of the CG location. Unfortunately, the term "X offset moment" is frequently described as "moment along X". This does not make mathematical sense, but like the term "pound mass" most engineers will understand the meaning.

Component distances for center of gravity location may be either positive or negative, and in fact their polarity depends on the choice of reference axis location.

The center of gravity of a homogeneous shape is calculated by determining the centroid of its volume. In real life, most objects are not homogeneous, so that the center of gravity must be computed by summing the offset moments along each of the three axes. These processes are described in detail in the following sections.

The center of gravity of an object can be located in "midair". For example, the center of gravity of a piece of pipe is on the centerline half way along its length, even though there is no metal in the center of the pipe (Figure 6). The composite CG of an object can be computed if the CG of each component is known. Examples follow.

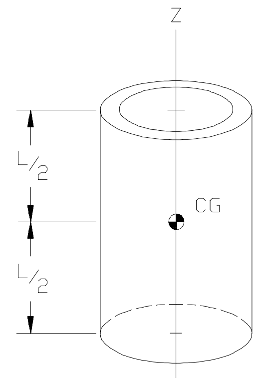


Figure 6

CG along a single axis

Consider the round metal rod with two cylindrical weights shown (Figure 7). Note that the elements do not have to be the same diameter to be symmetrical along the length. In fact the elements could overlap (such as sliding one pipe inside another). From symmetry, the CG of the object is on its centerline (since the CG of a homogeneous mass is at its centroid of volume). The CG location along the length can be determined by summing moments about the reference axis at the bottom of the figure (x = 0).

Assume that the element weights are; W_a=12.250 lb, W_b=4.613 lb, W_c=2.553 lb.

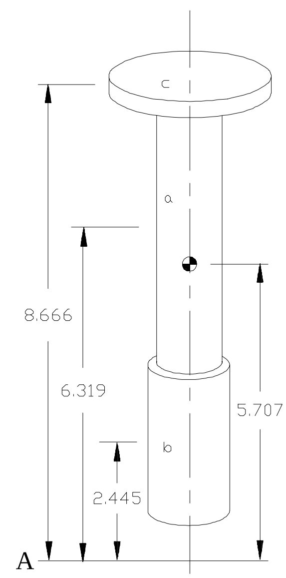


Figure 7 – CG along a single axis

$$M_a = W_a z_a \quad M_a = 12.250\, lbs\ x\ 6.319\ inch = 77.408\ lb-in$$

$$M_b = W_b z_b \quad M_b = 4.613\, lbs\ x\ 2.445\, inch = 11.279\ lb-in$$

$$M_c = W_c z_c \quad M_c = 2.553\, lbs\ x\ 8.666\ inch = 22.124\ lb-in$$

$$Total\ Weight = 19.416\ lbs$$

$$Total\ \ Moment = 110.811\ lb-in$$

$$\underline{CG}_z = \frac{Total\ \ Moment}{Total\ Weight} = \frac{110.811\ lb-in}{19.416\ lb} = 5.707\ inch\ \ from\ A$$

CG of Unsymmetrical Three Dimensional Body

The center of gravity of an unsymmetrical body may be calculated in the same manner as the single axis example above. Each axis may be considered separately (Figure 8).

Consider the cylinder with attached rectangles. The CG of each component is known by symmetry, computation, or measurement. A convenient frame of reference is assigned, in this case such that the CGs of each component fall on the axes, and offset moments are summed along each axis. Dimensions shown are to the CG of each component from the origin.

$$M_x = M_a + M_b + M_c = 0.4(1.75) + 0 + 0 = 0.7\ lb - in$$
$$\underline{CG}_x = 0.70\ lb - in / 4.8\ lb = 0.146\ in$$

$$M_y = M_a + M_b + M_c = 0 + 0 + 1.8(-1.25) = -2.25 lb - in$$
$$\underline{CG}_y = -2.25 lb - in / 4.8 lb = -0.469 in$$

$$M_z = M_a + M_b + M_c = 0.4(4.25) + 2.6(2.5) + 1.8(0.5) = 9.1\ lb - in$$
$$\underline{CG}_z = 9.1 lb - in / 4.8 lb = 1.896 in$$

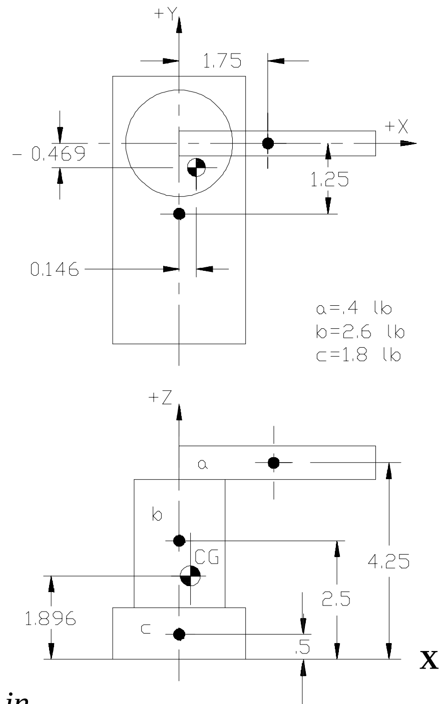


Figure 8 – CG of an unsymmetrical body

CG of a Complex Shape Similar to a Standard Shape

Consider the hollow cone shown below. From symmetry, the CG lies along the center line. The CG distance along the length could be calculated using calculus. However, the CG of a solid cone is given in the SAWE handbook. Using the observation that a hollow cone can be created by removing a small solid cone from a larger one, we can calculate the CG by subtracting the moment due to the smaller cone from the larger one. Volume moments are taken around the center of the base to find the centroid of the hollow cone. When the cone is combined with other elements to find the overall CG, its actual weight and calculated centroid location are combined with those of the other elements.

$$V_1 = \frac{\pi\, r_1^2 H_1}{3} = \frac{\pi\,(4^2)(20)}{3} = 335.1 \qquad V_2 = \frac{\pi\, r_2^2 H_2}{3} = \frac{\pi(3.6^2)(18)}{3} = 244.3$$

$$V_{NET} = V_1 - V_2 = 90.8\ in^3$$

$$\underline{CG} = \frac{a_1 V_1 - a_2 V_2}{V_{NET}}$$

$$\underline{CG} = \frac{(5)(335.1) - (4.5)(244.3)}{90.8} = 6.35\ in$$

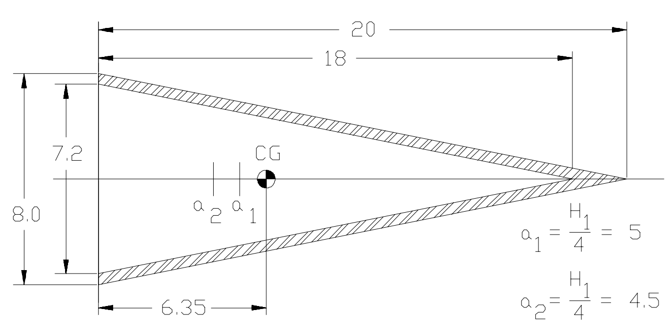
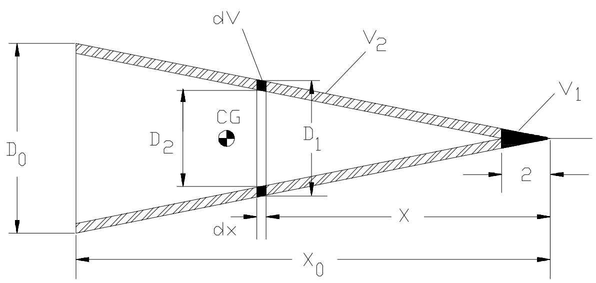


Figure 9 – Hollow Cone

CG of a Complex Unusual Shape

If you encounter a shape which is not in the handbook and which cannot be created from known shapes, then it will be necessary to use calculus to calculate its CG. The basic concept of the calculation is the same as the previous examples, except the moments that are summed are moments involving a small differential slice of the object rather than moments of discrete objects. The trick to simplifying this process is to chose the right differential shape, so that triple integration can be avoided. Your differential element should not be a small cube unless there is no symmetry of any kind. Generally you can use a rectangular bar that covers the full length of the part, or a thin disc or annular ring whose diameter is a function of location.

 To illustrate the CG calculation using calculus, we will use the same hollow cone discussed in the previous section.



$$D_o = 8'' \qquad X_o = 20''$$

$$V_T = \int_{x-0}^{x-2} dV_1 + \int_{x-2}^{x-20} dV_2$$

$$M_T = \int_{x-0}^{x-2} x\,dV_1 + \int_{x-2}^{x-20} x\,dV_2$$

$$dV_1 = \frac{\pi D_1^2}{4} dx \qquad D_1 = \frac{D_o}{X_o} X \qquad dV_2 = \frac{\pi}{4}\left(D_1^2 - D_2^2\right)dx \qquad D_2 = \frac{D_o}{X_o}(x-2)$$

$$dV_1 = \frac{\pi D_0^2}{4 X_0^2} x^2 dx \qquad dV_2 = \frac{\pi}{4}\frac{D_0^2}{X_0^2}\left[x^2 - \left(x^2 - 4x + 4\right)\right]dx$$

$$V_T = \frac{\pi D_0{}^2}{4 X_0{}^2}\left[\int_0^2 x^2 dx + \int_2^{20} (4x-4)dx\right] = \frac{\pi D_0{}^2}{4 X_0{}^2}\left[\Big|_0^2\left[\frac{x^3}{3}\right] + \Big|_2^{20}\left[\frac{4x^2}{2} - 4x\right]\right] = 90.8 in^3$$

$$M_T = \frac{\pi D_0{}^2}{4 X_0{}^2}\left[\int_0^2 x^3 dx + \int_2^{20} \left(4x^2 - 4x\right)dx\right] = \frac{\pi D_0{}^2}{4 X_0{}^2}\left[\Big|_0^2\left[\frac{x^4}{4}\right] + \Big|_2^{20}\left[\frac{4x^3}{3} - \frac{4x^2}{2}\right]\right] = 1240.0 in^4$$

$$Location\ of\ \ Centroid\ \ from\ tip = \frac{M_T}{V_T} = \frac{1240.0 in^4}{90.8 in^3} = 13.65$$

$$OR \qquad 20 - 13.65 = 6.35\ inch\ from\ Base$$

Rectangular to Polar Conversion
When first calculated, the CG data is in rectangular form. Often it is useful to convert this data into polar coordinates. Most computers and scientific calculators will do this automatically. However, if one is not available, the following method can be used (two axis):

Magnitude $M = \sqrt{X^2 + Y^2}$

Angle

$A = \text{arcTAN}(Y/X)$	if X = (+) and Y = (+)	(1st quadrant)
$A = 180^o - \text{arcTAN}(Y/X)$	if X = (-) and Y = (+)	(2nd quadrant)
$A = 180^o + \text{arcTAN}(Y/X)$	if X = (-) and Y = (-)	(3rd quadrant)
$A = 360^o - \text{arcTAN}(Y/X)$	if X = (+) and Y = (-)	(4th quadrant)

Polar to Rectangular Conversion
After the data has been converted to polar form, sometimes you may then need to convert it back to rectangular form, using a different set of axes. This might occur if you wanted to adjust the CG offset of a reentry vehicle, so that it was balanced about its centerline. In this case, the available locations for correction weights would usually not fall on the reference axes.

Step 1 Add appropriate offsets to the X and Y rectangular components.

Step 2 Convert to polar form (magnitude and angle).

Step 3 Add the appropriate offset angle to rotate the vector to the new X-Y coordinate system.

Step 4 Convert the new vector to rectangular coordinates using the following formula:

$$X_1 = M\cos(A_1) \qquad Y_1 = M\sin(A_1)$$

where X_1 and Y_1 are the new axes and A_1 is the angle between the unbalance moment vector and the X_1 axis.

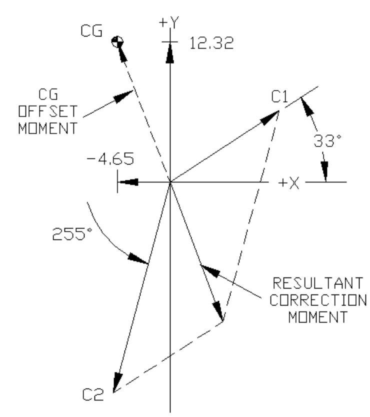


Figure 7 – Static Unbalance

Correcting Static Unbalance
The satellite shown has a calculated static unbalance (CG offset moment) of X= -4.65 lb-inch and Y = +12.32 lb-inch. It is necessary to add ballast weights to the vehicle so that this unbalance is reduced to zero. If weights could be added at 0^o and 270^o, then we would need 4.65 lb-in at 0^o to compensate for the -4.65 lb-in X axis unbalance and 12.32 lb-in at 270^o to compensate for the +12.32 lb-in Y axis unbalance. However, these locations are not available. This is the general situation in most aerospace balancing. In this example, the only locations where these

weights can be added are at 33° and 255°. The radius of the correction weight at 33° is 8.25 inches and at 255° is 7.60 inches. What weights should be added to each location to compensate for the unbalance?

The following example outlines the method used to determine the new weights at the allowed locations:

We will first calculate the resultant polar magnitude and angle, then calculate rectangular coordinates of correction moments. We will then divide the correction moments by their radii to obtain correction weights.

General Correction Equations

$$Sum\left[C\sin A_c\right] = M\sin(180 - A) \qquad Sum\left[C\cos A_c\right] = M\cos(180 - A)$$

where: C = Correction Moments

A_C = Allowable Correction Angles
M = Static unbalance Moment
A = Angle of Unbalance Moment

Note that these calculations involving static unbalance are concerned with weight rather than mass. In our example above, the figure did not show the height at which the weights were to be added. In general, the weights should be added at a height that is as close as possible to the CG height of the vehicle, so that the addition of these weights will not produce a large product of inertia unbalance.

Calculating Correction Weights

$$M = \sqrt{M_x{}^2 + M_y{}^2} = 13.17\ lb - in$$

$$A = TAN^{-1}(y / x)$$

$$A = TAN^{-1}(12.32 / -4.65) = 110.7^o$$

To find Correction Moments C_1, and C_2 at 33° and 255° locations:

$$C_x = C_1\,Cos33 + C_2\,Cos255 = 4.65$$

$$C_y = C_1\,Sin33 + C_2\,Sin255 = -12.32$$

$$.84C_1 - .26C_2 = 4.65$$

$$.54C_1 - .97C_2 = -12.32$$

Multiply both sides of eq. (2) by: -(0.84/0.54) and add to eq. (1)

$$0C_1 + 1.25C_2 = 23.81$$
$$C_2 = 19.05\ lb - in$$

$$C_1 = (4.65 + .26C_2)/.84$$
$$C_1 = 11.43\ lb - in$$

Weights to be added:

$$W_1 = \frac{11.43\,lb - in}{8.25in} = 1.39\,lb$$

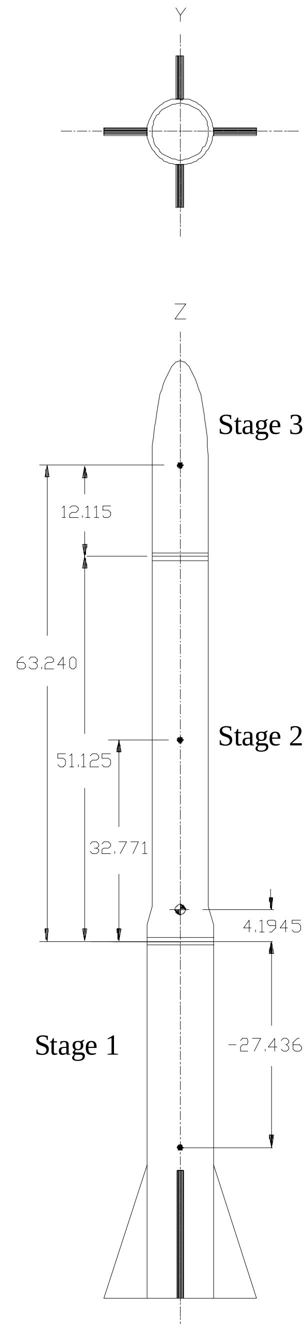
$$W_2 = \frac{19.05\,lb - in}{7.6in} = 2.51\,lb$$

Note that total weight is more than twice as large as would be required if ballast weights could have been added at the angle of the unbalance at 8" rad.

Combining CG Data from Subassemblies

Let us consider the case of a three stage rocket. The center of gravity of the individual stages had been originally calculated to be on the centerline. After construction, the CG of each section has been measured and found to be:

	X	Y	Z	W
Stage 1	+0.004"	-0.012"	-27.436"	167 lb.
Stage 2	-0.007"	+0.012"	+32.771"	96 lb.
Stage 3	-0.004"	+0.012"	+12.115"	43 lb.
			TOTAL	306 lb.



Two views of the rocket are shown. The top view shows the X and Y axes; side view shows X and Z axes.

The X and Y coordinates are measured from the centerline of the section; the Z coordinates of stages 1 and 2 are measured from their intersection, while the Z coordinate of the third stage is measured from the intersection of stages 2 and 3. In order to calculate the Z center of gravity location, we will first have to translate the stage 3 coordinate to the same reference as stages 1 and 2. Since the length of stage 2 is 51.125 inches, the Z coordinate becomes 12.115 + 51.125 = 63.240 inches.

Figure 8 – CG of Three Stage Rocket

If the three stages were perfectly aligned at assembly, then the combined CG of the total rocket could be calculated by summing X, Y, and Z offset moments about the origin:

	X	Y	Z
Stage 1	+0.668 lb-in.	-2.004 lb-in.	-4581.812 lb-in.
Stage 2	-0.672 lb-in.	+1.152 lb-in.	+3146.016 lb-in.
Stage 3	-0.172 lb-in.	+0.516 lb-in.	+2719.320 lb-in.
Total Moment	-0.176 lb-in.	-0.336 lb-in.	+1283.524 lb-in.
Moment/306 lb	-0.00058 in.	-0.00110 in.	+4.1945 in.

If the three stages are assembled with an alignment error, then:

1. Select one of the stages to be the reference. For this example we will choose stage 2. The CG coordinates for this stage will therefore remain unchanged.

2. Recalculate the CG coordinates for stages 1 and 3 to reflect the alignment error. If the stages did not assemble tightly along their length, so that there is a 0.006 inch gap between stages 2 and 3, then the 63.240 Z dimension becomes 63.246. If the X axis is shifted sideways on the first stage by +0.003 inch, then the -0.004 inch X dimension for the first stage becomes -0.001 inch, etc. If the stages are tilted relative to each other, then the offset due to the tilt must be determined at the CG height of the stage. For example, if stage 3 is tilted so that the error on the Y axis at a Z dimension of 24.5 inch is +0.020, then the Z axis correction for stage three is: .020 x 12.115/24.500 = 0.00989 inch.

The new value of Y for stage three is therefore: Y = 0.012 + 0.00989 = 0.0219 inch

3. After the revised table for the CG coordinates for stages 1 and 3 are complete, then the calculation proceeds in a manner identical to the example with perfect alignment.

CALCULATING MOMENT OF INERTIA

General Comments

Moment of inertia ("MOI") is similar to inertia, except it applies to rotation rather than linear motion. Inertia is the tendency of an object to remain at rest or to continue moving in a straight line at the same velocity. Inertia can be thought of as another word for mass. Moment of inertia is, therefore, rotational mass. Unlike inertia, MOI also depends on the distribution of mass in an object. The greater the distance the mass is from the center of rotation, the greater the moment of inertia.

A formula analogous to Newton's second law of motion can be written for rotation:

$F = Ma$ (F = force; M = mass; a = linear acceleration)
$T = IA$ (T = torque; I = moment of inertia; A = rotational acceleration)

Choosing the Reference Axis Location

Three reference axes were necessary to define center of gravity. Only one axis is necessary to define moment of inertia. Although any axis can be chosen as a reference, it is generally desirable to choose the axis of rotation of the object. If the object is mounted on bearings, then this axis is defined by the centerline of the bearings. If the object flies in space, then this axis is a "principal axis" (axis passing through the center of gravity and oriented such that the product of inertia about this axis is zero (see discussion of product of inertia). If the reference axis will be used to calculate moment of inertia of a complex shape, choose an axis of symmetry to simplify the calculation. This axis can later on be translated to another axis if desired, using the rules outlined in the section entitled "Parallel Axis Theorem".

Polarity of Moment of Inertia

Values for center of gravity can be either positive or negative, and in fact their polarity depends on the choice of reference axis location. Values for moment of inertia can only be positive, just as mass can only be positive.

Units of Moment of Inertia

In the United States, the word "pound" is often misused to describe both mass and weight. If the unit of weight is the pound, then the unit of mass cannot also be a pound, since this would violate Newton's second law. However, for reasons which have been lost in antiquity, in the USA an object weighing 1 pound is often referred to as having a mass of 1 pound. This leads to units of moment of inertia such as lb-in^2, where the "lb" refers to the weight of the object rather than its mass. Correct units of moment of inertia (or product of inertia) are: MASS x DISTANCE^2

When lb-in^2 or lb-ft^2 are used to define MOI or POI, the quantity MUST be divided by the appropriate value of "g" to be dimensionally correct in engineering calculations. Again, dimensional analysis will confirm if correct units are being used.

The following table shows some of the units in use today for moment of inertia and product of inertia:

UNIT	COMMENTS
lb-in^2	lb = weight; must be divided by g = 386.088 in/sec^2
lb-in-sec^2	lb-in-sec^2 = distance^2 x weight/g; weight/g = mass; dimensionally correct
slug-ft^2	slug = mass; dimensionally correct
kg-m^2	Kg = mass; dimensionally correct

The most common units used in the U.S. are lb-in^2 , even though this is dimensionally incorrect.

RULE 1. If moment of inertia or product of inertia are expressed in the following units, then their values can be used in engineering calculations as they are:

Slug-ft^2, lb-in-sec^2, kg-m^2, lb-ft-sec^2, oz-in-sec^2

RULE 2. If moment of inertia or product of inertia are expressed in the following units, then their values must be divided by the appropriate value of "g" to make them dimensionally correct.

lb-ft^2, lb-in^2, oz-in^2

Value of g : 32.17405 ft/sec^2 or 386.088 in/sec^2
Do not use local value of g to convert to mass!

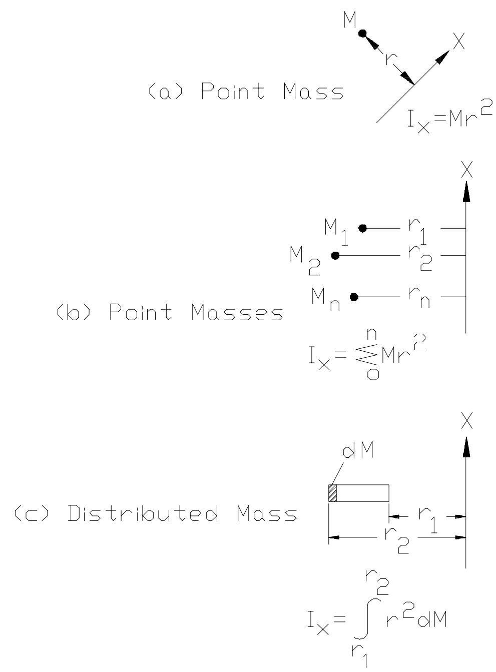


Figure 9

Calculating the Moment of Inertia
MOI, sometimes called the second moment, for a point mass around any axis is: $I = Mr^2$

where I = MOI (slug-ft^2 or other mass-length^2 units)
M = mass of element (Slugs or other mass unit)
r = distance from the point mass to the reference axis

Radius of Gyration
The moment of inertia of any object about an axis through its CG can be expressed by the formula: $I = Mk^2$

where I = moment of inertia
M = mass (slug) or other correct unit of mass
k = length (radius of gyration) (ft) or any other unit of length

The distance (k) is called the Radius of Gyration. The method of calculating radius of gyration is outlined in the following sections.

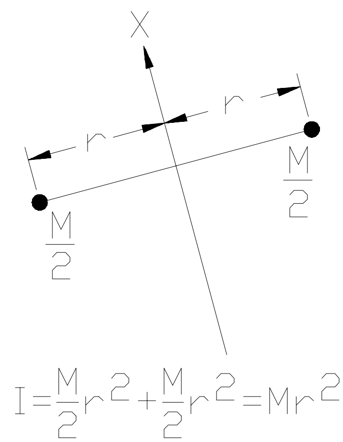


Figure 10

Consider first the body consisting of two point masses each with a mass of M/2 separated by a distance of 2r. The reference axis is through a point equidistant from the two masses. The masses each have a MOI of $Mr^2/2$. Their combined MOI is therefore Mr^2. The second example shows a thin walled tube of radius r. By symmetry, the CG lies on the centerline of the tube. Again, all the mass is located at a distance r from the reference axis so its MOI = Mr^2. In these examples, the radius of gyration is $k = r$. This leads to the definition:

"The radius of gyration of an object, with respect to an axis through the CG, is the distance from the axis at which all of the mass of an object could be concentrated without changing its moment of inertia. Radius of gyration is always measured from CG."

Parallel Axis Theorem

If in the example above we wanted to determine the MOI of the object about the axis X_a rather than the axis X, through the CG, then the value can be determined using the parallel axis theorem:

$I_a = I + d^2 M$,   Since $I = k^2 M$, then $I_a = M (d^2 + k^2)$

where k is the radius of gyration.

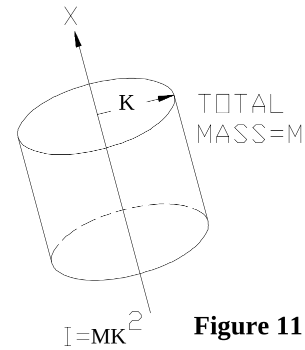


Figure 11

This parallel axis theorem is used very frequently when calculating the MOI of a rocket or other aerospace item. The MOI of each component in the rocket is first measured or calculated around an axis through its CG, and the parallel axis theorem is then used to determine the MOI of the total vehicle with these components mounted in their proper location. The offset "d" is the distance from the CG of the component to the centerline of the rocket.

Useful Approximations

Since the moment of inertia of an object displaced from its reference axis is proportional to $(d^2 + k^2)$, we can make two observations that will simplify the job of calculating MOI:

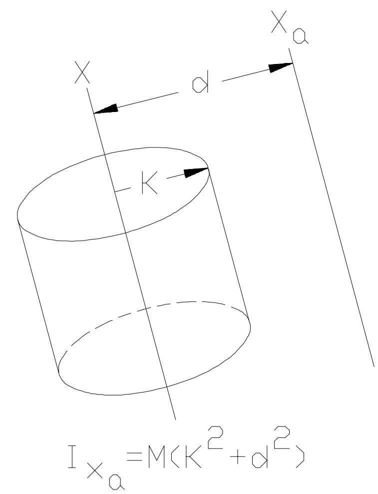


Figure 12

RULE 1. If the radius of gyration of an object is less than 1% of its offset distance "d", then the MOI of the object around its CG can be ignored when calculating total MOI, and the value becomes d^2M. For example if a gyro with a mass of 0.1 slug is located near the outer surface of a rocket and the offset to the CG of the gyro is 3 feet while the radius of gyration of the gyro is only 0.02 ft, then the MOI about the center line of the rocket due to the gyro is d^2M = 0.9 slug-ft^2. The error using this approximation is less than 0.01%.

RULE 2. If the radius of gyration of an object is more than 100 times its offset distance "d", then the offset of the object can be ignored when calculating total MOI, and the value becomes k^2 M. For example if a rocket motor with a mass of 100 pounds is located near the center line of the rocket and the offset to the CG of the rocket motor is 0.100 inches, while the radius of gyration of the rocket motor is 12 inches, then the MOI about the center line of the rocket due to the rocket motor is k^2 M = 14400 lb-in^2 (or more properly 37.3 lb-in-sec^2). Again the error of approximation is less than 0.01%

Rule 2 can also be applied to alignment errors when calculating or measuring MOI. If the offset or misalignment is less than 1% of the radius of gyration, then the alignment error is insignificant.

Combining Moment of Inertia of Two Objects

If the object contains more than one mass, then the moment of inertia is the sum of the individual moments of inertia taken about the same axis. The radius of gyration is:

$$k = \sqrt{\left(\frac{I_{total}}{M_{total}}\right)}$$

The moment of inertia of the two examples (fig. 13) is the same. Note that it makes no difference what angle the masses have relative to each other. Radius is the only factor affecting their moment of inertia.

These examples illustrate that moment of inertia depends only on the radius of the masses within an object. However, if the object were flying in space, since the CG, radius of gyration, and principle axis would be different for the two examples, their flight characteristics would differ.

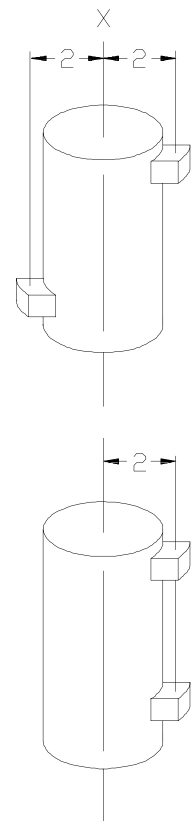


Figure 13

Each weight = 1 lb.

Radius from X axis=2 in.

$I_x = 1 \times 2^2 + 1 \times 2^2 = 8$ lb-in^2

The MOI about the X axis is the same for both examples

Basic Formula Using Differential Elements of Mass

The basic technique for calculating moment of inertia of an object is to consider each element of mass and its radius, apply the formula $I = Mr^2$ to each, and then add up all the moments of inertia of the elements.

If this were done as described, then the computation would be of the form:

$I_1 = M_1 (r_1)^2$ where r_1 is the radius of M_1, etc.

$I_2 = M_2 (r_2)^2$

$I_n = M_n (r_n)^2$ etc.

I = Total of above

If the object is a homogeneous solid, then this process can be accomplished by choosing a suitable differential element and integrating over the limits of the radius:

$$I = \int r^2 dM$$

Combining axial MOI values

If the axial moment of inertia of two cylindrical rocket sections about their mutual centerline are 10 slug-ft^2 and 20 slug-ft^2 respectively, then the total moment of inertia of both sections when assembled is 30 slug-ft^2. Moment of inertia values are simply added to obtain the total. Before adding the values, make certain that they are both calculated about axes which are coincident when assembled and that the units for each are consistent and correct. Alignment is relatively unimportant. The moment of inertia error due to misalignment is proportional to the ratio of the square of the misalignment offset to the square of the radius of gyration of the object. For example, if a rocket has a radius of gyration of 15 inches, and it is laterally misaligned by 0.002 inch, then the resulting error is only 0.000,002% (2 millionths of 1 percent)!

Combining Transverse MOI Values

The combination of MOI around transverse axes is a more complex procedure.

1. The MOI of each component around an axis through its own CG parallel to the desired axis must be determined by computation or measurement

2. The location of the composite CG must be calculated

3. The MOI of each component around the composite CG must be calculated using the parallel axis theorem

4. The MOIs are added to find the total MOI around the desired axis through the composite CG.

Standard Shapes

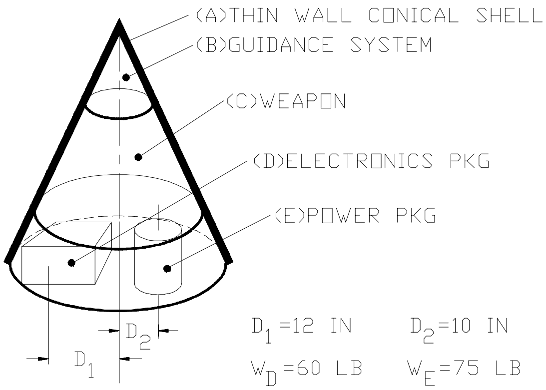
The moment of inertia of a variety of standard shapes has been published in most of the textbooks and handbooks that cover dynamics such as the SAWE Handbook, Machinery's Handbook etc.

Moment of Inertia of Objects Similar to Standard Shapes

Since total moment of inertia can be calculated by simply summing the values of the component parts, it is possible to derive the MOI of many shapes by modifying the values for standard shapes. This often eliminates the need for calculus and greatly speeds up the calculation of MOI. For example to determine MOI for a hollow cone, note that MOI of the inner conical space can be subtracted from the outer as a technique for simplifying calculations even though there is no such thing as negative moment of inertia.

Composite MOI Example

An example finding composite MOI around the Z (longitudinal) axis is shown using a reentry vehicle consisting of a hollow cone and other components. The parallel axis theorem is used to calculate I_Z for off center components.



$$I_A = \frac{WK^2}{g} \qquad W = 200\,lb \qquad K = 15\,in$$

$$I_B = 450\ lb - in^2$$

$$I_C = 25\,Slug - Ft^2$$

$$I_D = 1.8\ lb - in - \sec^2$$

$$I_E = 3.5\ lb - in - \sec^2$$

CALCULATE MOI AROUND Z AXIS

$$I_A = \frac{200\ lb(15)^2\ in^2\ ft^2\ sec^2}{144\ in^2\ 32ft} = 9.6\ lb\text{-}ft\text{-}sec^2 = 9.6\ slug\text{-}ft^2$$

$$I_B = \frac{450\ lb\text{-}in^2\ sec^2\ ft}{386.088\ in\ \ 12\ in} = 0.097\ lb\text{-}ft\text{-}sec^2 = 0\ .1\ Slug\text{-}ft^2$$

$Ic = 25\ Slug\text{-}ft^2$

$$I_{D_o} = \frac{1.8lb - in - \sec^2 - ft}{12in} = 0.15Slug - ft^2 \qquad I_{D_z} = .15 + \left(\frac{60}{32}\right)\left(\frac{12}{12}\right)^2 = 2.03slug - ft^2$$

$$I_{E_o} = \frac{3.5lb - in - \sec^2 - ft}{12\,in} = 0.29\ Slug - ft^2 \qquad I_{E_{(z)}} = .29 + \left(\frac{75}{32}\right)\left(\frac{10}{12}\right)^2 = 1.92\,Slug - ft^2$$

$Iz\ TOTAL = 9.6 + .1 + 25 + 2.03 + 1.92 = 38.65\ Slug\text{-}Ft^2$

Effects of Misalignment

When misalignment results in tilt errors, or if other effects cause the CG as well as MOI to change, more complex analysis must be performed. This will be discussed further under "Product of Inertia".

Calculating Product of Inertia

General Comments

Consider the homogeneous balanced cylinder to which two equal weights have been attached 180^{o} apart, and spaced equidistant along the length from the CG of the cylinder. The addition of these weights will not alter the CG of the cylinder, and the cylinder remains statically balanced. However, if we spin this cylinder about the vertical Z axis, then centrifugal force acts through the two weights and produces a couple. If the cylinder is mounted on bearings, then this couple causes a sinusoidal force to be exerted against the bearings as the cylinder rotates. If the cylinder is spinning in space, then the axis of rotation of the cylinder shifts to align itself to a condition where the centrifugal forces are equalized (i.e., it shifts toward the unbalance weights slightly). The mass distribution which results in a couple moment when the object is spinning is called "product of inertia."

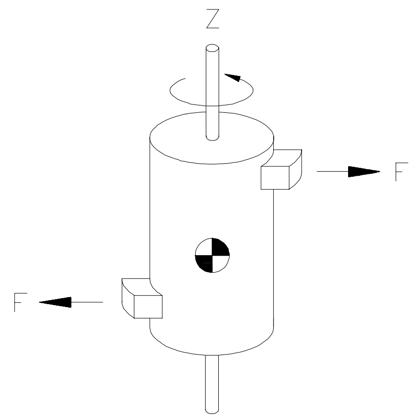


Figure 14– CGz=0, but weights cause couple moment

Basically, product of inertia ("POI") is a measure of dynamic unbalance. POI is expressed in the same units as moment of inertia, but it can have either a positive or negative polarity. Product of inertia is generally not covered in undergraduate dynamics courses, and consequently many engineers are unfamiliar with this concept.

Choosing the Reference Axis Location

Like center of gravity, three mutually perpendicular reference axes are necessary to define products of inertia (only one axis is necessary to define moment of inertia). Although any axis can be chosen as a reference, it is generally desirable to select the axis of rotation of the object as one axis. If the object is mounted on bearings, then this axis is defined by the centerline of the bearings. If the object flies in space, then this axis is often defined by the location of thrusters. On reentry vehicles, the axis may be coincident with the path of flight resulting from axis of symmetry of the outer surface of the vehicle. If the reference axis will be used to calculate product of inertia of a complex shape, choose an axis of symmetry to simplify the calculation. This axis can later on be translated to another axis if desired, using the rules outlined in the section entitled "POI Parallel Axis Theorem."

Polarity of Product of Inertia

Values for product of inertia can be either positive or negative, and in fact their polarity depends on the choice of reference axis location. In this respect, POI is similar to CG. Values for moment of inertia can only be positive, just as mass can only be positive. Generally, the product of inertia of one component is offset by a negative product of inertia due to another component, so that the composite product of inertia of a composite object will be much smaller than the product of inertia of many of its elements.

Units of Product of Inertia

Product of inertia is expressed in the units of mass times distance squared. For CG calculations we used the weight of the object; product of inertia calculations use the mass. Unfortunately, the word "pound" can mean either weight or mass, so the engineer must use caution when applying product of inertia values to engineering equations. (See section entitled “Units of Moment of Inertia”).

Sometimes engineers will encounter bogus units for product of inertia such as "oz-inch." Although such units are incorrect, they have meaning in context with the machine used to measure dynamic unbalance. This machine may have a readout for a specific correction plane height which gives the moment required at this plane height to reduce the product of inertia to zero. This data can be converted to valid units by multiplying the "oz-inch" moment by the height between the correction plane and the test object CG (and then converting the weight in ounces to a unit of mass). This is explained in more detail in the section on correction of dynamic unbalance.

Principal Axis

On any object there will be three mutually perpendicular axes intersecting at the CG for which the products of inertia will be zero. For a perfect cylinder, these axes correspond to the centerline of the cylinder plus two mutually perpendicular axes through the CG at any orientation (since the cylinder has perfect symmetry). These axes are called the "principal axes." The moment of inertia of the object is at a maximum about one principal axis and at a minimum about another principal axis. A spin stabilized vehicle will rotate about a principal axis (usually the axis of minimum moment of inertia).

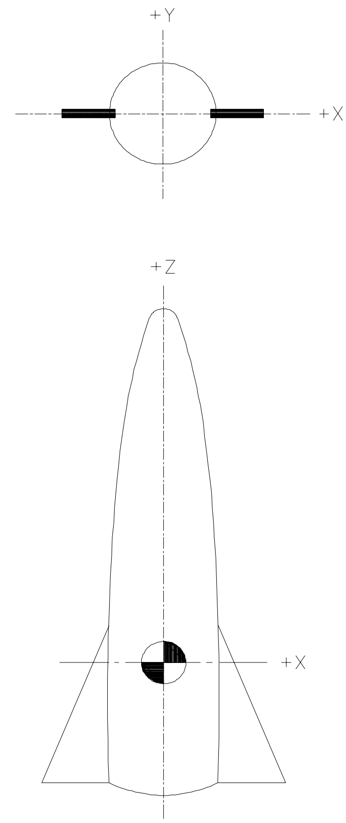


Figure 15 - X, Y and Z are principal axes

Calculating Product of Inertia

A perfectly balanced cylinder rotates on a set of bearings. A small weight whose POI is zero is mounted on this cylinder. The product of inertia due to this weight is:

P_{ZX} = M Z X = 0.01 x 2 x 1 = 0.02 slug-ft^2

where M = mass of weight = 0.01 slug

X = radius of CG of weight = 1 foot

Z = height between CG of cylinder & CG of weight = 2 feet

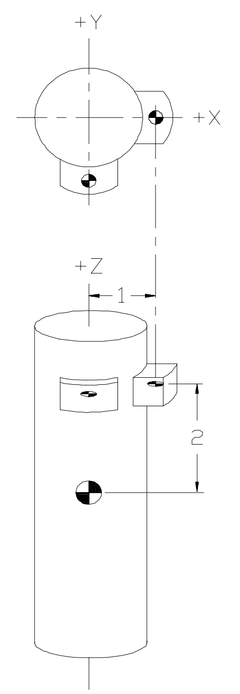


Figure 16 - calculation of P_{ZX} & P_{Zy}

This calculated POI is in the X-Z plane of the cylinder. If a similar weight were then added on the Y axis at a location above the CG of the cylinder, the value for P_{zx} would not change, since the X coordinate of this weight would be zero. The second POI component, P_{zy}, could be calculated as shown below.

The product of inertia due to this weight is: P_{zy} = MZY = 0.01 x 2 x (-1) = -0.02 slug-ft^2

where M = mass of weight = 0.01 slug
Y = radius of CG of weight = -1 foot
Z = height between CG of cylinder & CG of weight = 2 feet

Note that the value for P_{zy} is negative.

Rectangular to Polar Conversion

The previous examples were for a special case where the unbalance was located directly on either the X or Y axis. When this occurs, then the mathematics is simplified, because the unbalance can be analyzed as a two-dimensional problem on a plane rather than a three-dimensional problem. A realistic object generally contains a couple unbalance that does not fall directly on any axis. However, this unbalance can be converted into rectangular components that fall directly on the axes, so that simplified calculations are possible

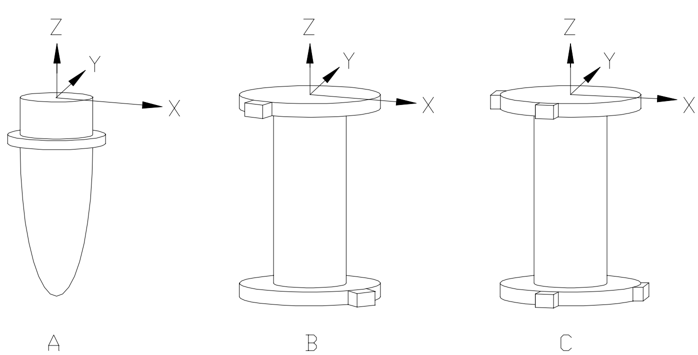


Figure 17 – Any POI can be simulated by two weights in upper plane and two weights in lower plane

The real object under test is shown in A. The unbalance equivalent of this object can be simulated by a single weight in each of the two planes as shown in B (angular difference between upper and lower planes can be any angle). As an aid to analysis, each single weight in the previous example can be replaced by two weights located on the X and Y axes as shown in C (i.e., polar to rectangular conversion). Each plane can now be analyzed separately.

At the conclusion of all calculations, the resulting P_{zx} and P_{zy} can then be converted back into polar coordinates if desired. The procedure for these rectangular-to-polar transformations is described in the section of this paper dealing with center of gravity.

The product of inertia of the two components in the previous example can be resolved into a single resultant P_{zr} in the ZR plane which passes through the equivalent unbalance mass and the Z axis.

$$P_{zr} = \sqrt{P_{zx}^{\ 2} + P_{zy}^{\ 2}}$$ Angle between resultant and X axis = arcTAN (P_{ZY} / P_{ZX})

Difference between CG Offset and Product of Inertia

The figures illustrate the difference between static unbalance (CG offset) and couple unbalance (product of inertia). In the example shown in figure 18, a 5 lb weight is added in the plane of the CG, creating a static unbalance but no product of inertia.

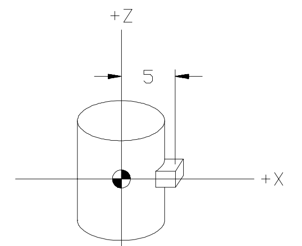


Figure 18- CG offset

P_{ZX} = 0 lb-in^2

$\underline{CG}_X$ = 25 lb-in

$\underline{CG}_Z$ = 0 lb-in

In the example shown in figure 19, a weight is added outside the plane of the CG, creating both static and dynamic unbalance (product of inertia not zero). (This is sometimes called "quasistatic unbalance," since a single correction weight can be used to correct for this unbalance.)

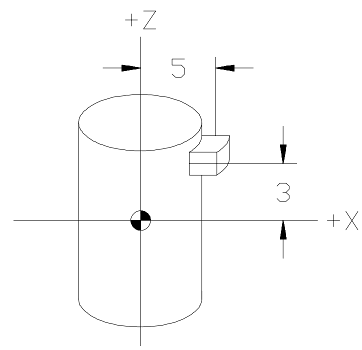


Figure 19 – Quasistatic unbalance

P_{ZX} = +75 lb-in^2

CG_X = +25 lb-in

CG_Z = +15 lb-in

In the example in figure 20, a second weight is added at 180^o at the bottom of the cylinder in the above example, creating a static balance but not a dynamic balance.

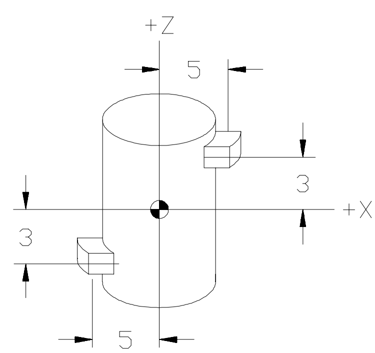


Figure 20 – Static & dynamic unbalance

P_{ZX} = 5 lb x 3 in x 5 in + 5 lb x (-3) x (-5) = 150 lb-in^2

CG_Z = +15 lb-in - 15 lb-in = 0

CG_X = +25 lb-in - 25 lb-in = 0

In figure 21, we move this lower weight up to the plane of the first, creating both static and dynamic balance about z.

P_{ZX} = +75 lb-in^2 - 75 lb-in^2 = 0

CG_Z = +15 + 15 = 30 lb-in

CG_X = +25 - 25 = 0 lb-in

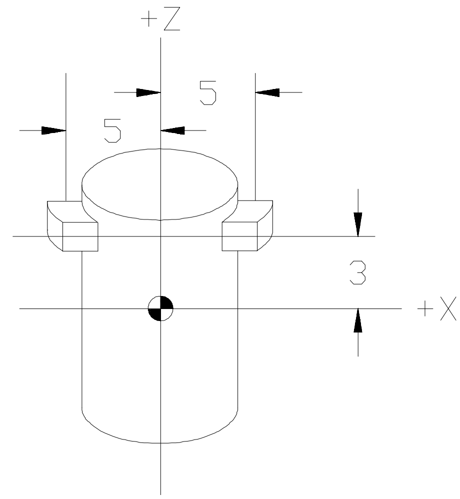


Figure 21 – Static & dynamic balance

The previous discussion has assumed that the small unbalance weights were perfectly symmetrical, and therefore the product of inertia of the weight itself could be ignored. In real life, the "weights" consist of various components of a rocket or spacecraft, and their product of inertia is usually not zero. Even if the component products of inertia are small, they cannot be ignored since the product of inertia of the total vehicle is usually very small, and even the smallest couple unbalance can tilt the principal axis of the vehicle.

POI Parallel Axis Theorem

When determining the product of inertia of a vehicle, it will be necessary to first calculate or measure the product of inertia of the component parts of the vehicle, and then translate these values to the effective POI about the axes of the vehicle.

To translate the product of inertia of an object relative to the X', Y', Z' axes to the X, Y, Z axes:

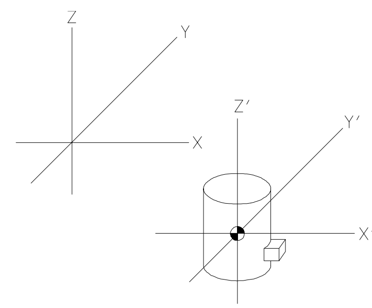


Figure 22– POI can be translated to another set of parallel axes

$$P_{zx} = P_{z'x'} + M\,z\,x$$

$$P_{zy} = P_{z'y'} + M\,z\,y$$

where M = mass of object
x, y, z = CG offsets from coordinate origin along the X, Y, Z axes

It is much more difficult to use this theorem than the equivalent one for moment of inertia, because there are two formulas required, and because each term has a polarity associated with it. The following example illustrates this type of translation:

Example for the illustration shown, let z = -4, x = +5, and y = -6 inches. The product of inertia of the object about the X', Y', Z' axes is $P_{Z'X'}$ = -2 lb-in-sec^2, $P_{Z'Y'}$ = 0 lb-in-sec^2. The weight of the object is 4 lbs. Calculate the effective product of inertia of the object relative to the X, Y, Z axes:

Calculate mass:

$$\frac{4}{(386.088)} = \frac{0.01036 lb - \sec^2}{in}$$

$$P_{zy} = P_{z'y'} + M\,z\,y$$

$$P_{zy} = 0 + (0.01036)(-4)(-6) = 0.24864\ lb - in - \sec^2$$

$$P_{zx} = P_{z'x'} + M\,z\,x$$

$$P_{zx} = -2 + (0.01036)(-4)(+5) = -2.2072\ lb - in - \sec^2$$

Comparison Between MOI and POI

There are some similarities and some differences between this axis translation formula and the formula to translate moment of inertia to a different parallel axis:

1. Both formulas are dimensionally similar:

(mass) $(\text{length})^2$ = (mass) $(\text{length})^2$ + (mass) $(\text{length})^2$

However, the polarity of the values of (mass) $(\text{length})^2$ can only be positive for moment of inertia, whereas they can be either positive or negative for product of inertia.

2. In the case of moment of inertia, it is possible to ignore the MOI of the object about its CG if the translation term is large. This is not so for product of inertia! If the POI of the object about its own CG is not zero, it cannot be ignored, even if the translation term value, Mxy, is large, since small values of product of inertia can be very significant if one large term is subtracted from another, leaving a small difference.

3. In the case of moment of inertia, the value for the MOI about the CG of an object always has a value greater than zero. It is possible for the product of inertia of an object to be zero, so that the translation formula becomes P_{ZX} = M z x.

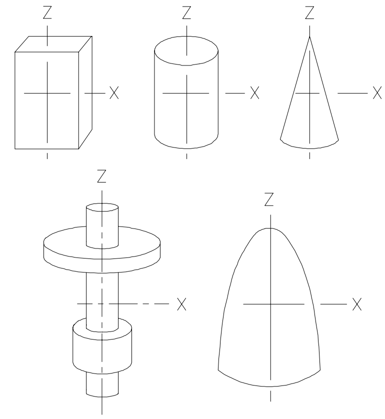


Figure 23- P_{zx} is zero for all of the examples above

Axes and Planes of Symmetry
The product of inertia of a homogenous body with respect to any pair of perpendicular axes is EQUAL TO ZERO if the plane determined by either of the axes and the third coordinate axis is a plane of symmetry of the body. This rule is hard to visualize when put into words. The examples on the right illustrate some symmetrical shapes that have a zero P_{ZX}.

Determining Product of Inertia of a Volume
The basic concept of determining the product of inertia of a homogeneous volume is identical to the previous method involving discrete objects, except the objects are now differential elements of a solid. The formula becomes:

$$P_{yx} = M\int yx dV$$

where M = total mass of object; dV = differential volume

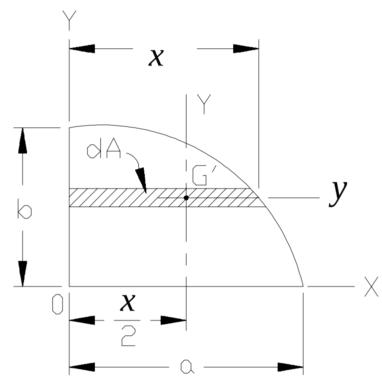


Figure 24 – Calculation of POI

As in the case for moment of inertia and center of gravity, the solution to the problem can be simplified by choosing the right differential element. For example, the elliptical wing tip shown can be analyzed using a small square element dX by dY. This leads to a double integral. If a rectangular slice parallel to the X axis is chosen instead, then the POI of the element is zero, and the product of inertia of dA is:

The CG of the rectangular slice is at x/2

$$(0 + 0.5\, x\, y\, dA)$$
$$Since\ dA = x\ dy$$
$$then \quad P_{zy} = 0.5\int x^2\ y\ dy$$

From the equation of an ellipse:

$$x^2 = \frac{a^2 b^2 - y^2 a^2}{b^2}$$

Therefore:

$$P_{xy} = 0.5\int \frac{a^2 b^2 - y^2 a^2}{b^2}\ y\ dy$$

Combining the POI of Two Bodies

If two sections of a rocket are combined, what is the resulting product of inertia? If the sections are aligned perfectly, so that the Z reference axis for the lower section is exactly coincident with the Z reference axis for the upper section, then the following method can be used:

1. Transform the product of inertia of the lower section into values for P_{ZX} and P_{ZY} (polar to rectangular conversion). If the X and Y axes for the upper section do not correspond to those chosen for the lower section, rotate the data by converting temporarily in polar form and then back into the new rectangular axes, so that upper and lower axes are at the same angular location. Analysis will be done by planes and transformed into polar coordinates if desired after all calculations have been made.

2. Sum the values for P_{ZX} upper and P_{ZX} lower. Do the same for P_{ZY}. Note: observe the polarity of the data! This will yield new product of inertia values for the composite vehicle. (The values for the total can be larger or smaller than the individual values.)

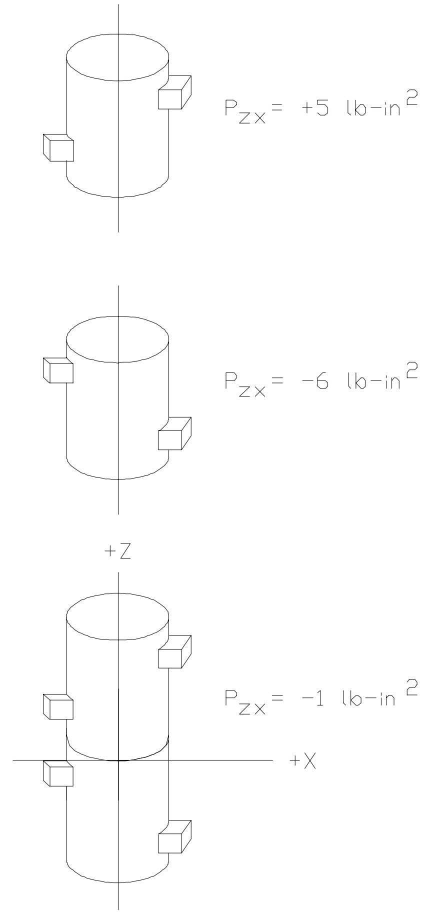


Figure 25 – Combining POI

Effect of Lateral Misalignment

What happens if the two sections are not aligned, so that the axes are parallel to each other, but the axis of one is not coincident with the axis of the other? Here is the method we recommend for analyzing this type of problem:

1. The location of the upper section must be defined in terms of the lower reference axis. This requires the measurement of offset of the upper section when both sections are assembled. This can be accomplished by placing the total rocket on a rotary table and dial indicating both lower and upper section; or if this is not possible, then measurements can be made of the individual sections and their interface ring concentricity and the offset calculated (not as accurate a method).

2. Calculate the X, Y, and Z coordinates of the CG of the total rocket. The new reference axis for the total rocket will pass through the new CG and be parallel to the reference axes of the two sections. The centerline of the two sections will be offset from this axis by some small but not insignificant amount. Often, the magnitude of POI resulting from misalignment will be greater than the POI of the individual sections, so that it is not satisfactory to ignore this effect. The X or Y axis distance will be small, but the mass will be very large since it is the entire mass of a section.

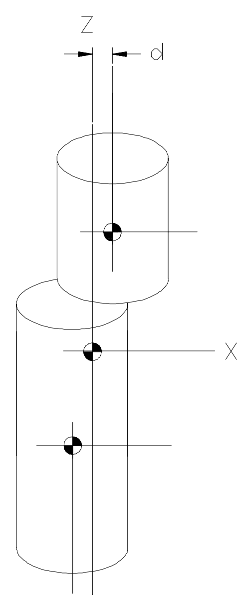


Figure 26 – Lateral misalignment

3. P_{ZX} relative to the new combined axis due to the P_{ZX} of the upper section may be calculated by using the parallel axis translation formula:

$$P_{zx} = P_{z'x'} + M\,z\,x$$

where P_{ZX} = POI relative to combined CG

$P_{Z'X'}$ = POI relative to upper section
M = mass of upper section x
z = distance between composite CG and upper section CG
x = offset between new combined reference and upper reference

In the view shown, both z and x are positive, so that the POI due to the upper offset is positive.

4. Repeat the calculation for the lower section. In the view shown, both z and x are negative, so that the POI due to the lower section offset is also positive.

5. Sum the values for P_{ZX} upper and P_{ZX} lower to yield P_{ZX} total.

6. Repeat Steps 3, 4, and 5 for P_{ZY} .

7. If desired, convert P_{ZY} and P_{ZX} into P_{ZR} , the resultant polar representation.

Effect of Tilt on MOI and POI

If a perfectly balanced cylinder is tilted by an angle "a," then P_{ZX}, I_Z, and I_X will change. As the cylinder leans to an angle of 90^o, I_Z becomes I_X, and I_X becomes I_Z. For a lean angle of 0^o, P_{ZX} is zero. For a lean angle of 45^o, P_{ZX} is a maximum determined by the values of I_Z and I_X. At a lean angle of 90^o, P_{ZX} is again zero. This unique relationship between moment of inertia and product of inertia is discussed in the SAWE paper No. 1473 entitled "Determining Product of Inertia Using a Torsion Pendulum." This paper outlines a method of measuring POI of objects using a moment of inertia instrument.

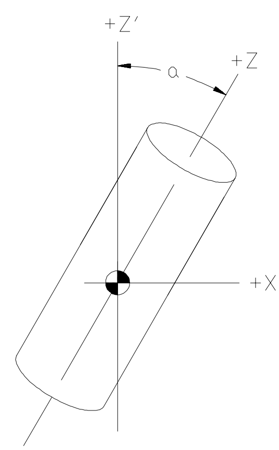


Figure 27– Axis Tilt

When the MOI or POI of an object is determined about its center line, and the object is then installed in a vehicle in such a way that there is a lean angle between the center line of the object and the reference of the vehicle, then it is very useful to be able to convert the calculated values for the object into mass properties relative to the new reference without having to recalculate the object itself. These "lean angle formulas" are given below.

MOI Inclined Axis Formulas

For the balanced cylinder shown, the moment of inertia about an axis Z' displaced from the center line of the cylinder by an angle "a":

$$I_{z'} = 0.5(I_z + I_x) + 0.5(I_z - I_x)\cos(2a)$$

Note that for this example, P_{ZX} was zero. This formula is only valid for the orientation of the axes shown. There are some interesting observations to be made regarding the change in MOI:

1. The MOI at an angle of 45^o is the average of I_Z and I_X.

2. The sensitivity to tilt angle is a function of the difference between I_X and I_Z. If there is very little difference, then tilt angle can be ignored. If the object is tall and slender, then tilt angle is very critical.

In addition to being useful in the calculation of MOI, this formula also can be used to determine fixturing accuracy required when measuring MOI. When measuring tall, slender rockets, the axial MOI error will be large unless the rocket is fixtured very carefully. Transverse MOI (I_y and I_X) can be measured on a vee block fixture without the need to adjust the rocket's position, since the sensitivity to lean is very small in this case.

MOI Incline Axis with POI

The previous analysis assumed that the Z and X axes were principal axes and the POI was zero. If this is not the case, then the formula becomes:

$$I_{z'} = 0.5(I_z + I_x) + 0.5(I_z - I_x)\cos(2a) + P_{zx}\sin(2a)$$

This formula reflects the fact that the principal axis is no longer through the centerline of the cylinder, so that the maximum and minimum MOI are no longer axes X and Z.

The formulas presented also assume that there is no tilt in the Y direction, so that the problem can be analyzed from a two-dimensional standpoint. If this is not the case, then the coordinate system must be transformed so that it is. Furthermore, note that the origin of the two axes is at the CG of the object. Equations can be written for the more general case. However, it is easier to manipulate the axes than to solve the general equations.

POI Inclined Axis Formulas

Since POI and MOI are related, it might be assumed that a similar formula can be written for the POI of an object when tilted. In this case, we are starting with a value of P_{ZX}, which is zero and returning to a value of zero at an angle of 90^o. The formula is:

$$P_{z'x'} = 0.5(I_x - I_z)\sin(2a) - P_{zx}\cos(2a)$$

If the X and Z axes are principal axes, then the formula becomes:

$$P_{z'x'} = 0.5(I_x - I_z)\sin(2a)$$

These formulas are only valid for the orientation of the axes shown (fig. 27) and for the direction and definition of positive tilt angle (CCW from the Z axis).

Mohr's Circle

A graphical representation of the relationship between MOI and POI was originated in the 19th century by a German engineer, Otto Mohr. A copy of this aid is reproduced from the SAWE handbook and is shown on the next page. With the advent of the personal computer, graphical solutions to engineering problems are no longer necessary; but Mohr's circle still is useful in visualizing the effect of tilt.

Mohr's Circle for Moments of Inertia

Given (1) The moment of inertia values I_X, I_Y for an object about its center of gravity, where the center of gravity lies at the origin of a set of mutually perpendicular axes X-Y.

(2) The corresponding value for the product of inertia, P_{XY}

Mohr's circle is then constructed using the layout geometry shown below. The following information may then be obtained.

(1) The location of the principal axes about which the moments of inertia are maximum and minimum and the products of inertia are zero.

(2) The corresponding maximum and minimum values of moments of inertia.

(3) The moments and products of inertia for any other set of mutually perpendicular axes A-B whose origin lies at the center of gravity of the given object and rotated C degrees from the original axes X-Y (reference, the figure to the right).

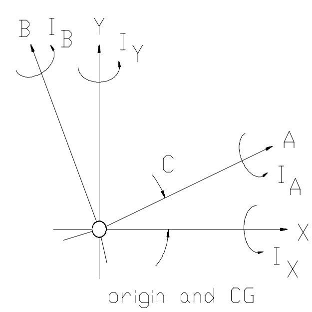


Figure 28

(4) The maximum values for the products of inertia about axes located 45^o from the principal axes.

Layout Geometry

$$\textit{The Radius of the circle is: } R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + \left(P_{xy}\right)^2}$$

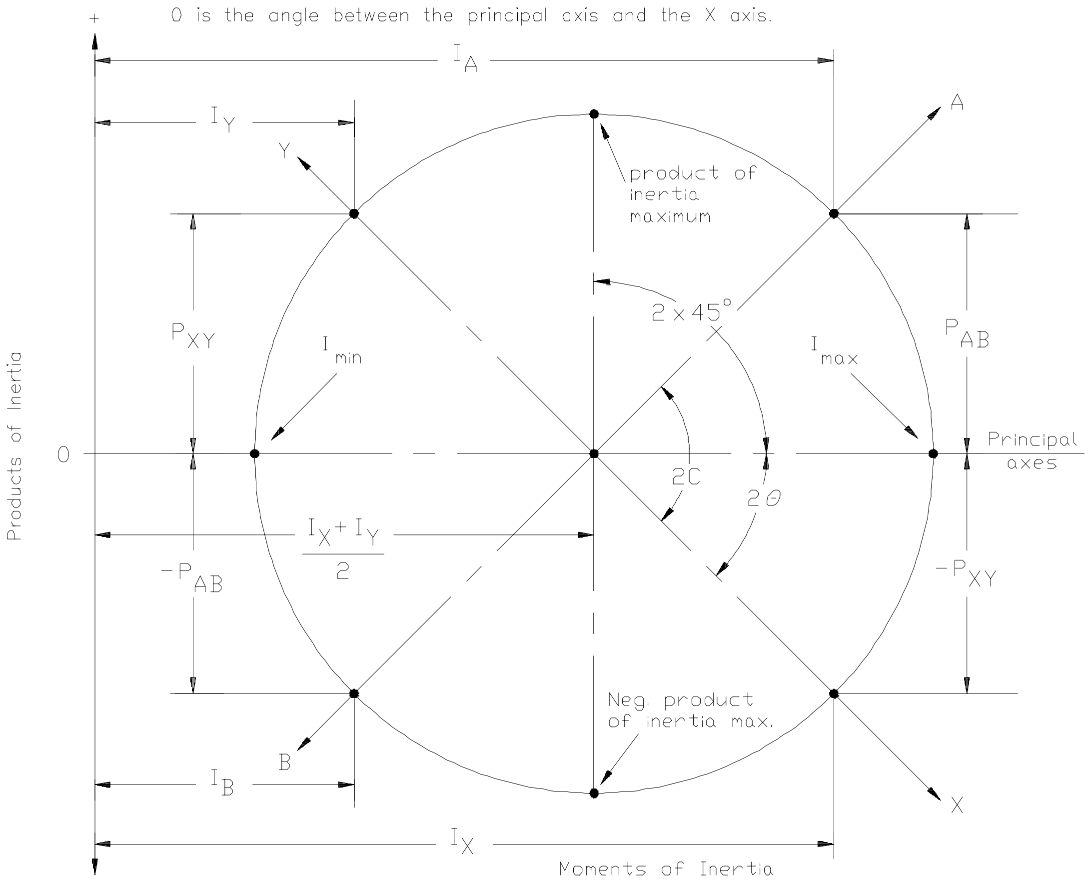


Figure 29 – Mohr's Circle

Effect of Angular Misalignment (Tilt)

If the upper section is tilted relative to the lower section, then two factors tend to increase the effective POI of this section: the tilt results in a CG offset similar to the case described previously, and the tilt also alters the POI of the upper section itself. The method for calculating the total POI is as follows:

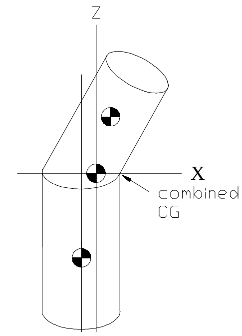


Figure 30 – Angular Misalignment

1. Using the center line of the lower section as a reference, calculate the Y axis offset of the CG of the upper section from the formula:

$$Y = H \sin a$$

Where: H is the CG height of upper section
a is the tilt angle in the z-y plane

Using a similar concept, calculate the X offset of the upper section.

2. Calculate the X, Y, and Z coordinates of the CG of the total rocket. The new reference axis for the total rocket will pass through the new CG and be parallel to the reference axes of the lower section.

3. Recalculate the P_{ZX} of the upper section by applying the axis tilt formula. Add this POI to the P_{ZX} of the upper section relative to its center line (observe signs; value for P_{ZX} may be either larger or smaller than value without considering tilt).

4. P_{ZX} relative to the new combined axis due to the P_{ZX} of the upper section may be calculated by using the parallel axis translation formula:

$$P_{zx} = P_{z'x'} + M z x$$

where P_{ZX} = POI relative to combined CG reference axes

$P_{Z'X'}$ = POI relative to upper section after effect of tilt has been added
M = mass of upper section
z = distance between composite CG and upper section CG
x = offset between new combined reference and upper reference

In the view shown, both z and x are positive, so that the POI due to the upper offset is positive.

5. Repeat the calculation in Step 4 for the lower section. Since this section is not tilted, the $P_{Z'X'}$ is the value through the center line. In the view shown, both z and x are negative, so that the POI due to the lower section offset is also positive.

6. Sum the values for P_{zx} upper and P_{zx} lower to yield P_{zx} total.

7. Repeat Steps 3, 4, 5 and 6 for P_{zy} .

8. If desired, convert P_{zx} and P_{zy} into P_{zr} , the resultant polar representation.

Angle of Inclination of Reentry Vehicle

An aerospace vehicle will often have an axis defined by the minimum air resistance of the vehicle. This axis corresponds to the axis of symmetry of the outer surface of the vehicle. Since the vehicle is not homogeneous, the product of inertia about this axis may not be zero, resulting in a principal axis at an angle to the axis of symmetry of the vehicle. The angle is known as the "angle of inclination" of the vehicle. Generally, it is desirable to make this angle as small as possible so the vehicle will "fly straight". Sometimes, however, this angle is deliberately adjusted to a specific value, so a reentry vehicle will have a "coning" motion upon entering the atmosphere, and the resulting drag will slow the reentry.

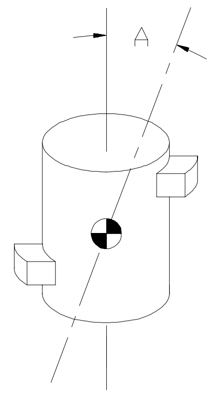


Figure 31 – Angle of Inclination

Angle of inclination may be calculated using the following formula:

$$A = 0.5arcTAN \frac{2P_{zx}}{(I_{zz} - I_{xx})}$$

Example: Given $P_{zx} = 0.002$ lb-in-sec^2

$I_{xx} = 8.95$ lb-in^2

$I_{zz} = 2.40$ lb-in^2

What is the angle of inclination in the X-Z plane? First, the units of moment of inertia must be converted to lb-in-sec^2 to be consistent with the units for product of inertia (or the units for product of inertia could be converted into lb-in^2):

$$I_{xx} = 8.95\,lb - in^2 = 0.023155\,lb - in - \sec^2$$

$$I_{zz} = 2.40lb - in^2 = 0.006216\,lb - in - \sec^2$$

Then, using the formula:

$$A = 0.5\,arcTAN\left[\frac{2P_{zx}}{(I_{zz} - I_{xx})}\right]$$

$$A = 0.5\,arcTan\left[\frac{0.004}{(0.006216 - 0.023155}\right] = -6.74\ \deg rees$$

The previous example was for a single plane. This would also be the angle of inclination for the object if the product of inertia of the Z-Y plane were zero. If there were a product of inertia for the Z-Y plane, then the combined solution for the object would be:

$$A = 0.5\,arcTAN\left[\frac{2P_{zr}}{(I_{zz} - I_{rr})}\right]$$

where P_{zr} is the resultant of P_{zx} and P_{zy}.

I_{rr} is the moment of inertia about axis "r" (resultant)

For a typical projectile or reentry vehicle, $I_{yy} = I_{xx}$, so that I_{xx} can be used in place of I_{rr}. For a vehicle with wings, this is not the case, and I_{rr} must be calculated from the values of I_{yy} and I_{xx}. (See section on moment of inertia.)

Controlling Tilt Angle

Since the amount of axis tilt which results from a given product of inertia is a function of the difference between I_{zz} and I_{xx}, the effect of a product unbalance can be adjusted by altering the difference in moment of inertia. This leads to two conclusions:

1. If you want to stabilize the vehicle and have it resistant to the effects of product unbalance, make the difference in moment of inertia as large as possible. This is accomplished by designing the vehicle to be long and slender, and by placing the heavy items near the ends of the vehicle.

2. If you want to steer the vehicle with as little correction force as possible, then make the moment of inertia difference as small as can be tolerated. The minimum moment of inertia difference will be limited by the skill with which you can correct product of inertia unbalance (a function of the sensitivity of the balancing machine used, and the stability of the components in the vehicle).

Unbalance of Rotating Objects

Unbalance Forces Due to Offset CG of Rotating Object

If a rotating object is mounted on bearings, then CG offset from the axis of rotation will produce a sinusoidal force on the bearings. This unbalance force will cause premature wear of the bearings, noise, additional friction, and may lead to errors if the rotating object is part of a guidance system.

The force exerted by a CG offset is a function of rotation speed and the magnitude of unbalance moment:

$$F = M\,r\,w^2$$

where F = unbalance force in pounds
M = mass in slugs = W/g
r = CG offset in feet
w = angular velocity in radians per second

If we examine the force exerted on a bearing system along a single radial axis, it varies sinusoidally:

$$F = Mrw^2 \sin a$$

where a = angle of rotation relative to the axis

Since a rotating system usually has two bearings, the force on each bearing would be a proportion of the total force (if the CG were equidistant from both bearings, then half the force would be applied to each bearing).

This analysis assumes that the product of inertia is zero. Such would be the case if the object were a thin flywheel, or if the object were dynamically balanced and a single weight was then added in a plane perpendicular to the axis of rotation and passing through the center of gravity of the object. If the product of inertia is not zero, then the forces on the bearings will be different from this example. The following section describes the forces due to product of inertia unbalance ("couple unbalance").

Unbalance Forces Due to POI of Rotating Object

For a CG offset, the forces on the bearings do not depend on the spacing between the bearings but do depend on the axial location of the CG of the rotor. If the rotor CG is located between the bearings, then the unbalance force is in the same direction for both upper and lower bearings; but the magnitude of the force is proportional to the CG location relative to the bearings and is, in general, different for each bearing. In contrast, a product of inertia unbalance will result in equal and opposite forces on the bearings and the force is not a function of the CG location. For a given POI, the magnitude of the bearing force will increase as the spacing between the bearings is made smaller.

One method of calculating the force due to product unbalance is to first determine the equivalent mass at the bearings that would result in the magnitude of POI. The force on the bearings can then be determined using the formulas given above for CG offset.

If a shaft with a P_{ZX} of 100 lb-in^2 is supported on bearings which are 10 inches apart, then the equivalent product of mass times distance at each bearing is 10 lb-in. If the rotation speed is 300 RPM, then:

$$300\,RPM = \frac{300}{60} = 5\,rev/\sec = 5\,x\,2\,x\,3.1416 = 31.416\,rad/\sec$$

$$10\,lb-in = \frac{10}{(32.174x12)} = 0.0259\,slug-\ ft$$

$$F = Mrw^2 = 0.0259\,slug-\ ft\ x\ (31.416)^2 = 25.56\ lbs\ force\ (peak)$$

These relationships form the basis for a spin balance machine. An analysis of centrifugal forces acting against the bearings of a spin balance machine results in measured values for center of gravity offset and product of inertia.

Mass vs Weight (and English vs Metric)

In 1999, the Mars Climate Orbiter crashed as a result of a confusion over the system of units. The software program which controlled the thrusters was supplied with thrust data in pound-seconds but interpreted it as if it were newton-seconds, result in an underestimation of the thruster impulse by a factor of 4.45. This is only one of many thousands of errors that have occurred as a result of the confusion between Metric and English units.

Many of these errors are the result of a misunderstanding regarding the difference between mass and weight. If you place an object on a scale in Europe, you will read its mass (generally expressed in *kg*). However, if you place an object on a scale in the USA, you will read a value equal to the force exerted by the acceleration of gravity (generally expressed in *lbf*). Since you are really trying to use the scale to measure mass, when you weigh yourself on an American bathroom scale, it should read 6.22 slug rather than 200 lbf.

Traditionally, a dimensionally inconsistent correction factor is used to convert from one set of units to the other. The expression 1 kg = 2.205 lb is not valid. It is like comparing apples to oranges. Mass does not equal force. This traditional conversion factor is based on the value of standard gravity, which is 9.80665 m/sec^2.

Mass is related to weight through Newton's second law: $$W = Mg$$

where W = the weight of the object (gravity force)
M = the mass of the object
g = the acceleration of gravity

MASS is the QUANTITY OF MATTER in an object (its inertia), while WEIGHT is the FORCE that presses the object down on a scale due to the acceleration of gravity. The mass of an object is a fixed quantity; its weight varies as a function of the acceleration of gravity. The mass properties of an object are related to mass, not weight. Mass properties do not change as a space vehicle leaves the attraction of the earth and enters outer space.

If different names are used for weight and mass, then the problem of distinguishing between the two is minimized. The Metric SI system uses the word "Newton" for weight and the word "Kilogram" for mass. The Newton is defined as the force required to accelerate a 1 Kilogram mass by 1 meter per second^2. The aerospace industry has created a unit of mass called the "Slug." A one pound force is required to accelerate a one Slug mass at one ft/sec^2. If an object weighs 32.17405 lbf on earth, then its mass is one Slug.

Unfortunately, not all systems of units adequately differentiate between mass and weight. In the USA, the word "pound" is commonly used for both mass and weight, resulting in endless confusion and errors in calculating mass properties and dynamic response. Officially “pound” refers to mass (see, for example, NIST documents). However, the common usage of the word pound is the value you read on a scale, which is actually *lbf*. If the term "pound" is used to describe a mass whose measured weight is one pound (force), this quantity MUST be divided by the acceleration of gravity in appropriate units to convert it to proper mass dimensions if it is to be used in mass properties calculations. Similarly, in metric countries the terms Kilogram and Gram are often, incorrectly, used to describe force as well as mass. To avoid confusion and uncertainty, an analysis of fundamental dimensions will confirm if correct units of measurement are being used and if conversion factors are being applied correctly to achieve desired results.

The various metric systems are fundamentally MASS, LENGTH, TIME systems with force being a defined or derived term. The U.S. systems are fundamentally FORCE, LENGTH, TIME systems with mass being defined or derived. Table One shows the three most commonly used systems of measurement. Time in seconds is used throughout

DIMENSIONALLY CORRECT MEASURING SYSTEMS

	MASS	LENGTH	WEIGHT	g
SI (Metric)	Kg	Meter	Newton	9.80665 M/sec^2
U.S. (inch)	$\frac{\text{Weight in lbf}}{\text{386.0886 in.}}$	Inch	Pound (lbf)	386.0886 in/sec^2
U.S. (foot)	Slug	Foot	Pound	32.17405 ft/sec^2

The U.S. inch system has no common name for the mass whose weight equals one pound, although this is sometimes called a "pound mass". One pound mass is equal to one pound force divided by 386.088 inches/sec2. Applying W = Mg shows that the system is dimensionally consistent.

$$1\,lb\,mass\;x\;g = \frac{1\,lb - \sec^2}{386.09\,in}\;x\;\frac{386.09\,in}{\sec^2} = 1\,lb\,weight$$

The acceleration of gravity used to convert weight to mass is a fixed number which has been established as an international standard.

Conversion Factors

Standard acceleration of gravity

g = 32.17405 ft/sec^2 or 9.80665 m/sec^2

Dimensionally inconsistent conversion factors based on standard acceleration of gravity

1 oz (ounce) = 28.349 52 gram
1 oz tr (troy ounce) = 31.103 48 gram
1 lbf (pound force) = 0.453 592 37 kg
1 kg = 2.204 622 6 lbf
1 kg = 9.806 65 Newtons
1 sh tn (short ton, US) = 907. 184 7 kg
1 ton (long ton, UK) = 1016. 047 kg

Force conversion factors

1 dyne = 10^{-5} N
1 lbf (pound-force) = 4.448 22 N
1 kp (kilopond) = 9.806 65 N

Mass conversion factors

1 kg = 0.068 521 76 lb mass
1 t (tonne, metric) = 68.521 76 lb mass
1 lb mass = 14.593 904 kg
1 t (tonne, metric) = 1000 kg

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